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ON THE EVALUATION OF OPTIMAL AND NON-OPTIMAL CONTROL STRATEGIES[*]

by

E. Polak[**]

INTRODUCTION

Since the conversion of a system from non-optimal or quasi-optimal opera-
tion is usually quite costly, it would be very desirable to be able to predict the
maximum probable improvement in the performance of the system, resulting from
such a conversion. This, of course, requires that the characteristics of the
system and the conditions under which it operates be entirely understood. However,
it is also necessary to have some definite measure of performance, one may be
called a figure of merit, which can be used for comparing the performance of
different systems. Such a figure of merit must be a number which expresses
some very significant aspect of the performance of the system in terms of the
cost function to be used for optimization. Thus, for example, in the case of
the midcourse guidance of a moonshot, a minimum fuel system may be proposed,
and, as a figure of merit one may choose the <u>average</u> estimated <u>fuel consumption</u>
for a trajectory or the <u>maximum</u> estimated fuel consumption for a trajectory.
It is clear from this example that more than one figure of merit may be associ-
ated with an optimization cost function and that all of them may be important.
Thus, in a complex system, different parts of which may have different cost
functions associated with them, the various figures of merit obtained can be
combined with figures expressing cost, reliability, etc., to form either a com-
pound and more realistic performance index, or to form a vector cost function[1].

If the system is a regulator, the most likely significant aspects of performance in terms of the optimal control cost function are average estimated cost and maximum cost for a single process within a specified class. These will be defined more explicity further on and methods for their computation, based on the properties of optimal controls, will be suggested.

DESCRIPTION OF A TYPICAL REGULATOR SYSTEM

Consider a system with a configuration as shown in fig. 1. and which is described by the differential equation:

where $\underline{x} = col(x_1, \ldots, x_n)$ is the state vector of the system, $\underline{u}(t) = col(u_1, \ldots, u_n)$ is the feedback computer output and $\underline{f}$ is a continuous and differentiable function in all its variables. The vector $\underline{u}(t)$ is constrained to lie within some set Ω which is closed and bounded. The system (1) may be sampled-data, in which case $\underline{u}(t)$ is subject to a PAM or PWM modulation law[2], and the state is observed at sampling instants only. The desired terminal state for the regulator will be denoted by $\underline{x}_d$ and the cost of taking the system from an initial state $\underline{x}_0$ to $\underline{x}_d$ will be denoted by $c(\underline{x}_0)$. The cost is usually defined by an integral evaluated along the trajectory in question and has the form

$$c(\underline{x}_0) = \int_0^T f^0(t, \underline{x}, \underline{u})dt \qquad (2)$$

where T is the time it takes to transfer $\underline{x}_0$ to $\underline{x}_d$ and f^0, the cost function, is continuous in all its variables.

Further, it is assumed that for each mode of operation there exists a closed bounded set $\mathfrak{X}$ in the state space, which contains all the possible initial states. Let us also assume that the probability $p(\underline{x})dV$ of an initial

state $\underline{x}$ lying in the element of volume dV of X is known. Initial states are usually caused by disturbances and these may be studied in an existing system. If the disturbances produce initial states independently of the control strategy used (as in magnetic memories where the address is the disturbance), then a single determination of the probability density function will suffice for all control laws under consideration. Otherwise, one may have to resort to simulation, using available data about the disturbances, in order to determine the initial states probability distribution as well as the set X on which it is defined.

For a system described as above, one may define two figures of merit:

$$m_1 = \int_{\underline{x} \in X} c(\underline{x}) p(\underline{x}) dV \tag{3}$$

which represents the average estimated cost of the process, and

$$m_2 = \sup_{\underline{x} \in X} c(\underline{x}) \tag{4}$$

which represents the maximum possible cost for a single transition inside the given set X and is in fact the maximum estimated cost for a single transition.

METHODS FOR EVALUATING m_1

There are obviously many ways in which a computation of m_1 can be carried out, however, only one which is very simple will be indicated here.

First, it is necessary to construct an ordering for any finite number of vectors in the set X. Let Q_m, $m=1, 2, \ldots, 2^n$ be the open quadrant defined by

$$Q_m = \left\{ \underline{x}: \ \underline{x} = \sum_{i=1}^{n} \lambda_i (-1)^{\alpha_i^m} \underline{b}_i, \ \lambda_i > 0, \ m=1, 2, \ldots, 2^n \right\} \tag{5}$$

where $\underline{b}_i = \mathrm{col}(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ is a unit vector whose i^{th} component is one and the rest are zero and the α_i^m are determined from the equation

$$m-1 = \sum_{i=0}^{n-1} \alpha_i^m 2^i, \ \alpha_i^m \{0, 1\} \tag{6}$$

Consider now any finite set of vectors $\chi^* \doteq \left\{ \underline{x}^1, \underline{x}^2, \ldots, \underline{x}^{\ell} \right\}$ in χ.

Let us agree to the convention:

(a) if $\underline{x}' \in \chi^* \cap Q_i$ and $\underline{x}'' \in \chi^* \cap Q_j$, and $i < j$, then $\underline{x}' > \underline{x}''$

(b) if $\underline{x}'$ and $\underline{x}''$ are both in $\chi^* \cap Q_m$ and if

   (i) $\left| x'_1 \right| > \left| x''_1 \right|$, or

   (ii) $x'_i = x''_i$ for $i = 1, 2, \ldots, k$, $k < n$ and

   $$\left| x'_{k+1} \right| > \left| x''_{k+1} \right|$$

(7)

then $\underline{x}' > \underline{x}''$.

By means of such an ordering, the vectors $(2, 0, 0, 1)$, $(2, 0, 1, 0)$, $(1, 3, 1, 2)$ and $(1, 2, 3, 4)$ would be arranged into a table in ascending order as shown below:

$$(1, 2, 3, 4)$$
$$(1, 3, 1, 2)$$
$$(2, 0, 0, 1)$$
$$(2, 0, 1, 0)$$

Now construct the smallest cube containing χ and, dividing each co-ordinate into $2k$ equal parts, divide this cube into $(2k)^n$ cubic elements ΔV_i with centres x_i. For all those centres x_i whose probability of being in ΔV_i is not zero, construct a table as shown below, using the ordering convention (7).

Table No. 1

$\underline{x}_i$	$p(\underline{x}_i) \, \Delta V_i$

in Table No. 1, $p(\underline{x}_i) \, \Delta V_i$ denotes the probability of $\underline{x}_i$ being in ΔV_i.

Case 1

If the control strategy is given in explicit form which is usually the case in non-optimal control, then one can evaluate the costs $c(\underline{x}_i)$ for the centres of the ΔV_i by direct integration and store them in a third column added to Table No. 1 to form Table No. 2.

Table No. 2

x_i	$p(x_i) \Delta V_i$	$c(x_i)$
	$\uparrow$	

Summing over Table No. 2, one may now obtain a first approximation for m_1:

$$m_1 = \sum_{i=1}^{(2k)^n} c(x_i)p(x_i) \, \Delta V_i \qquad (8)$$

In order to test the goodness of such an approximation, one may refine the partition of the cube and repeat the above operations. If the change in the approximate value of m_1 is sufficiently small, say 5 percent, one may stop, otherwise one may have to continue refining the partitioning of the cube.

Case 2

If the control law is not given in an explicit form, which is frequenctly the case in optimal control, but is given as an iterative algorithm, or as an algorithm involving the storage of switching surfaces, then it may be more convenient to take advantage of the known structural properties of optimal controls. Thus, if one can represent each optimal control as a point in a parameter space, one can find the corresponding initial state by solving the plant differential equation backwards in time starting from the point x_d. As an example consider the minimum time second order regulator

$$\underline{x} = \underline{A}\,\underline{x} + \underline{d}u(t), \quad |u| \leq 1 \qquad (9)$$

where $\underline{A}$ is a 2 x 2 constant matrix with real eigenvalues and $\underline{d}$ is a constant vector. The optimal controls for this system have the form shown in fig. 2 from which it is clear that the two parameters t_1 and t_2 describe an optimal control up to its polarity. Thus, each point (t_1, t_2) of the parameter plane maps into two points.

$$\underline{x}' = e^{-(t_1+t_2)\underline{A}} \underline{x}_d + \int_0^{t_1} e^{-\tau\underline{A}} \underline{d} \, d\tau - \int_{t_1}^{t_1+t_2} e^{-\tau\underline{A}} \underline{d} \, d\tau$$

$$\underline{x}'' = e^{-(t_1+t_2)\underline{A}} \underline{x}_d - \int_0^{t_1} e^{-\tau\underline{A}} \underline{d} \, d\tau + \int_{t_1}^{t_1+t_2} e^{-\tau\underline{A}} \underline{d} \, d\tau$$

Now, devise a suitable grid in the parameter space, such as shown in fig. 3 for the second order minimum time regulator, and map the grid points into the state space, thus obtaining points $\underline{x}'_i$ with known costs $c(\underline{x}'_i)$. Now, still using the ordering (7), construct Table No. 3 by entering the $\underline{x}'_i$ and $c(\underline{x}'_i)$ into Table No. 1.

Table No. 3

$\underline{x}_i$ and $\underline{x}'_i$	$p(\underline{x}_i) \; \Delta V_i$	$c(\underline{x}'_i)$

In Table No. 3, most rows will have entries only either in the first and second column or in the first and third column. Let $\underline{x}_{ij}, j=1, 2, \ldots, k_i$, be the cube centres lying in Table No. 3 between the point $\underline{x}'_i$ and $\underline{x}'_{i+1}$, then a first approximation to the figure of merit m_1 will be given by expression

$$m_1 = \sum_i c(\underline{x}'_i) \left(\sum_j p(\underline{x}_{ij}) \Delta V_j \right) \tag{9}$$

More accurate approximations can then be obtained by refining the grid in the parameter space and further partitioning of the elementary cubes ΔV_i.

METHODS FOR EVALUATING m_2

It is reasonably clear that the tables prepared for evaluating m_1 can be scanned to find an approximate value for m_2. It should be observed, however, that the cost function $c(\underline{x})$ may be discontinuous, as in the case in

minimal time control of PAM and PWM regulator systems, and in such a case one would have to allow a margin in the positive direction on the value of m_2, of magnitude equal to the size of the largest jump in the value of $c(\underline{x})$ over X. For the mentioned PAM and PWM minimum time systems this is exactly one unit.

CONCLUSION

It might be interesting to point out that parametrization of the optimal control space is possible in a number of important cases such as minimal time and minimal fuel control of continuous and discrete regulator systems when the plants are linear or when they can be shown to be optimal strategy equivalent to certain linear plants[2].

In conclusion, the author would like to remark that there was no intention to minimize the difficulties involved in properly identifying the plant and the nature of the disturbances. However, these must be treated on a case by case basis.

REFERENCES

1. Zadeh, L. A. "Opimality and Non-Scalar-Valued Performance Criteria," IEEE Trans., Vol. AC-8, No. 1, Jan. 1963, pp. 59-60.

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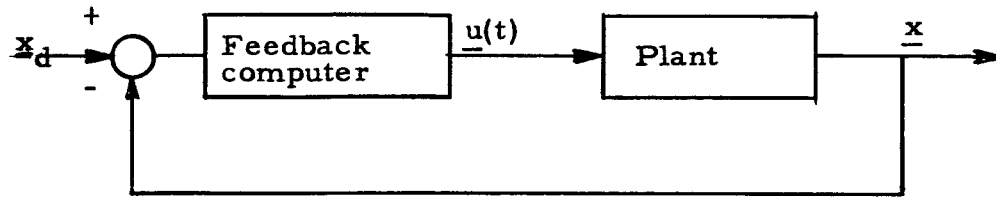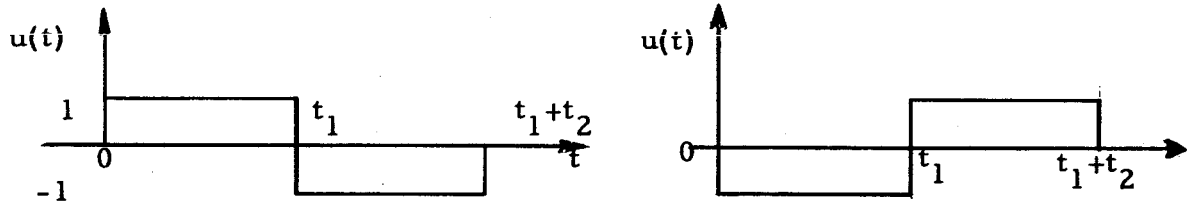
FIG. 1.   REGULATOR CONFIGURATION

FIG. 2.   OPTIMAL CONTROLS FOR SECOND ORDER REGULATOR
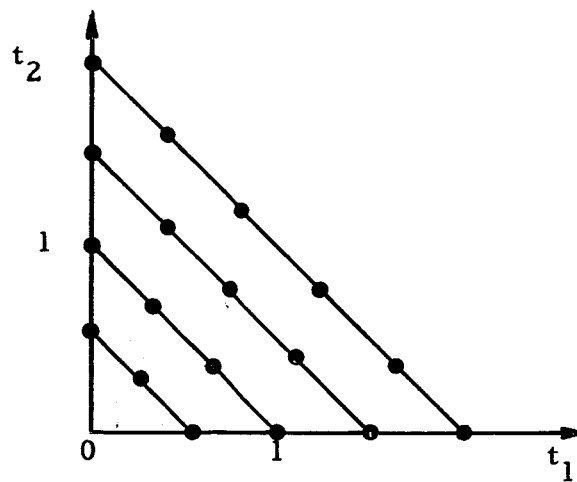WITH REAL EIGENVALUES.

FIG. 3.   PARAMETRIZATION OF CONTROL SPACE FOR SECOND
ORDER REGULATOR WITH REAL EIGENVALUES.